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ON THE PERMANENT AXES OF ROTATION OF A GYROSTAT WITH A FIXED POINT

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Permanant rotations of a heavy rigid body were discovered by Mlodzeevskii [1] and Staude [2].

The necessary conditions for the stability of permanent rotations of a heavy rigid body were investigated by Grammel [3]. The sufficient conditions for stability of permanent rotations both for a general case with arbitrary mass distribution inside the body, and for a number of special cases were derived by Rumiantsev [4]. A detailed investigation of permanent rotations of a gyrostat moving by inertia, and of its stability is due to Volterra [5]. A geometrical interpretation of the motion of a gyrostat in the latter case was given for the first time by Zhukovskii [6]. The problem of distribution of permanent axes of rotation of a heavy gyrostat has been partially solved by Anchev [7] and Drofa [8]. The necessary and sufficient conditions of stability for certain motions of heavy gyrostats were found by Rumiantsev [9].

In this work we determine the permanent axes of rotation of a gyrostat under the action of forces resulting from a force function U, and depending only on the position of the gyrostat.

We assume that the gyrostat S consists of the rigid body S_1, having a fixed point O and of the bodies S_2 joined nonpermanently with S_1. The angular momentum of the bodies S_2 in their motion with respect to the body S_1 is assumed to be constant. We shall investigate the stability of certain motions of the gyrostat using the second method of Liapunov.

1. The orientation of the rectangular axes $OXYZ$ determine the position of the gyrostat S with the fixed point O. The axes $OXYZ$ are fixed in the body S_1 and coincide with the principal axes of inertia of the gyrostat S moving about its fixed point. The orientation of $OXYZ$ refers to the rectangular coordinate system $O\xi\eta\zeta$ fixed in space. Let $\gamma_1, \gamma_2, \gamma_3$ be the direction cosines of the ζ-axis with respect to moving axes X, Y, Z; let A, B, C be the principal moments of inertia of the gyrostat S for its point O; let p, q, r be the X, Y, Z components of the instantaneous angular velocity of the body S_1. If the angular momentum $\mathbf{k}$ of the relative motion of the bodies S_2 is constant and if U is a function of $\gamma_1, \gamma_2, \gamma_3$ only, then the motion of the gyrostat is described by the system of six equations

$$A\frac{dp}{dt} + (C-B)qr + qk_3 - rk_2 = \frac{\partial U}{\partial \gamma_2}\gamma_3 - \frac{\partial U}{\partial \gamma_3}\gamma_2 \qquad \begin{pmatrix} p, & q, & r \\ A & B & C \\ 1, & 2, & 3 \end{pmatrix} \tag{1.1}$$

$$\frac{d\gamma_1}{dt} = r\gamma_2 - q\gamma_3 \tag{1.2}$$

Here k_1, k_2, k_3 are the X, Y, Z components of the vector $\mathbf{k}$; the remaining equations are obtained by circular permutations of the indices displayed inside parentheses.

Equations (1.1) and (1.2) permit the three first integrals

$$W_1 = Ap^2 + Bq^2 + Cr^2 - 2U = \text{const}$$

$$W_2 = (Ap + k_1)\gamma_1 + (Bq + k_2)\gamma_2 + (Cr + k_3)\gamma_3 = \text{const}, \qquad W_3 = \gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$$

2. Reasoning as the author in [1] we shall find out that the gyrostat can rotate permanently with a constant angular velocity about the fixed axis ζ. Let the direction cosines of the permanent axis with respect to the XYZ axes be denoted a, b, c, and let the XYZ components of the vector ω (ω = const) be written as

$$p = \omega a, \qquad q = \omega b, \qquad r = \omega c \tag{2.1}$$

Equations (1.1) then take the form

$$\omega^2 (C-B) bc + \omega (bk_3 - ck_2) = \beta_2 c - \beta_3 b \qquad (ABC,\ abc,\ 123)$$

$$\beta_i = \left(\frac{\partial U}{\partial \gamma_i}\right)_{\gamma_i = a, b, c} \qquad (i = 1, 2, 3) \tag{2.2}$$

and Equations (1.2) are satisfied identically. The system of equations (2.2) together with the relation

$$a^2 + b^2 + c^2 = 1 \tag{2.3}$$

can be used for determination of ω, a, b, c. Let us multiply Equations (2.2) by a, b, c, respectively, and add them. The sum equals identically zero which means that every quadruple ω, a, b, c which satisfies any pair of Equations (2.2) must satisfy the third eqaution as well. Consequently we shall consider only two of the equations in the system (2.2), for example the first and the second one, assuming at the beginning

$$A \neq B \neq C \tag{2.4}$$

$$\omega^2 - 2D_1\omega + E_1 = 0, \qquad \omega_2 - 2D_2\omega + E_2 = 0 \tag{2.5}$$

$$D_1 = \left(-\frac{1}{2}\right)\frac{bk_3 - ck_2}{(C-B)bc}, \qquad D_2 = \left(-\frac{1}{2}\right)\frac{ck_1 - ak_3}{(A-C)ac}$$

$$a \neq 0, \qquad b \neq 0, \qquad c \neq 0 \tag{2.6}$$

$$E_1 = (-1)\frac{\beta_2 c - \beta_3 b}{(C-B)bc}, \qquad E_2 = (-1)\frac{\beta_3 a - \beta_1 c}{(A-C)ac} \tag{2.7}$$

For every triple a, b, c we have two quadratic equations for ω and they are equivalent only when

$$D_1 \equiv D_2, \qquad E_1 \equiv E_2 \tag{2.8}$$

However, the relations (2.4) do not imply (2.8). The necessary and sufficient condition for Equations (2.5) to coincide is that their resultant

$$(E_2 - E_1)^2 - 4(D_2 - D_1)(D_1E_2 - E_1D_2) = 0 \tag{2.9}$$

must vanish.

Equation (2.9) in variables a, b, c has the form

$$\{ab\,[\beta_3(A-B)] + bc\,[(B-C)\beta_1] + ac\,[\beta_2(C-A)]\}^2 + \{ab\,[k_3(A-B)] + bc\,[k_1(C-B)] + ac\,[k_2(A-C)]\}\,\{a\,[\beta_2 k_3 - \beta_3 k_2] + b\,[\beta_3 k_1 - \beta_1 k_3] + c\,[\beta_1 k_2 - k_1\beta_2]\} = 0 \qquad (2.10)$$

and it determines the locus of the permanent axes of the gyrostat. If we pass a unit sphere with its center at the point O, then the locus of the points of intersection of the surface (2.10) with the sphere will be a certain curve on the sphere. A line joining any point of this spherical curve with the origin can be a permanent axis, if the angular velocity ω as determined from Equation (2.5) for this line, is real. The points on this spherical curve possessing this property will be called permissible.

For example let $U = \text{const}$, then the locus of the permanent axes is the second order cone

$$bc\,[k_1(C-B)] + ac\,[k_2(A-C)] + ab\,[k_3(B-A)] = 0$$

All the points of the spherical curve and the generating lines of the cone are permissible, and the angular velocity of the permanent rotation is found from

$$\omega = -\frac{bk_3 - ck_2}{(C-B)\,bc} \qquad (2.11)$$

If the force function is

$$U = mg\,(x_0\gamma_1 + y_0\gamma_2 + z_0\gamma_3) + \frac{3g}{2R}\,(A\gamma^2 + B\gamma_2^2 + C\gamma_3^2)$$

where (x_0, y_0, z_0) are the coordinates of the center of gravity, R is the distance between the center of attraction and the point O, then the locus of the permanent axes is the surface

$$mg\,\{ab\,[z_0(A-B)] + bc\,[x_0(B-C)] + ac\,[y_0(C-A)]\}^2 + 3gR^{-1}\{ab\,[k_3(A-B)] + bc\,[k_1(B-C)] + ac\,[k_2(C-A)]\}^2 - mg\,\{a\,[y_0k_3 - z_0k_2] + b\,[z_0k_1 - x_0k_3] + c\,[x_0k_2 - y_0k_1]\} \times \{ab\,[k_3(A-B)] + bc\,[k_1(B-C)] + ac\,[k_2(C-A)]\} = 0$$

If $U = \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_3\gamma_3$ and if for example $A > B > C$, and $\beta_1 > 0$, $\beta_2 > 0$, $\beta_3 > 0$, then the locus of the permanent axes becomes a fourth order surface. The equation of this surface has the form (2.10) where $\beta_1, \beta_2, \beta_3 = \text{const}$.

Let $\operatorname{grad} U \neq \lambda\mathbf{k}$. The surface (2.10) has the following generators: 1) the principal axes of inertia X, Y, Z; (2) the line between the points (0, 0, 0) and $(\beta_1, \beta_2, \beta_3)$. The points of intersection of these generators with the sphere will be denoted respectively by X^+, Y^+, Z^+, G^+, and diametrically opposite points by X^-, Y^-, Z^-, G^-.

Let the solid angles made by the half-planes passing through the above mentioned points be

$$\Theta_1 = \{X^+Y^+X^-,\ X^+G^+X^-\}, \qquad \Theta_2 = \{X^+G^+X^-,\ X^+X^-Z^+\}$$
$$\Theta_3 = \{X^+X^-Z^+,\ X^+X^-Y^-\}, \qquad \Theta_4 = \{X^+X^-Y^-,\ X^+X^-G^-\}$$
$$\Theta_5 = \{X^+X^-G^-,\ X^+X^-Z^-\}, \qquad \Theta_6 = \{X^+X^-Z^-,\ X^+X^-Y^+\}$$

The angular velocity is then either $\omega = D_1 + \sqrt{D_1^2 - E_1}$, or $\omega = D_1 - \sqrt{D_1^2 - E_1}$. Consequently the points of the spherical curve, contained in the angles $\Theta_1, \Theta_3, \Theta_5$ are permissible if $E_1 < 0$; the points of the spherical curve contained in the angles $\Theta_2, \Theta_4, \Theta_6$ would not be permissible when $D_1 = 0$, and could be permissible if $D_1^2 - E_1 \geqslant 0$.

We shall study now the problem of finding the angular velocity of gyrostat's rotation about the permanent axis $l\{a, b, c\}$.

The l-axis is a permanent axis if and only if the numbers a, b, c satisfy Equation (2.10) which means that they must satisfy at least one of the four relations

$$D_1 \pm \sqrt{D_1^2 - E_1} = D_2 \pm \sqrt{D_2^2 - E_2} \tag{2.12}$$

If the numbers a, b, c satisfy one of the relations (2.12) and besides $D_1^2 - E_1 > 0$, then the rotation about l with angular velocity ω can indeed occur, and ω would be equal to the left member of the corresponding equation from (2.12). It is easily seen that if the numbers a, b, c satisfy only two of the relations (2.12), then either $D_1 = D_2$ and $E_1 = E_2$, or $D_1^2 = E_1 = 0$ or $D_2^2 - E_2 = 0$. In the first case the gyrostat can rotate about the l-axis in opposite directions with different angular velocities $\omega_1 = D_1 + \sqrt{D_1^2 - E_1}$ and $\omega_2 = D_1 - \sqrt{D_1^2 - E_1}$. In the second case the angular velocity about the l-axis equals ω_1 or ω_2, respectively. It can be shown that if the numbers a, b, c satisfy three of the (2.12) relations, then they must also satisfy the fourth one. In this case the rotation with angular velocity $\omega = D_1$ about the permanent axis $l\{a, b, c\}$ is possible. One can make similar considerations without assuming (2.4) and (2.6).

3. We shall investigate the stability of permanent rotations of a gyrostat assuming that U is a twice differentiable function of γ_1, γ_2, γ_3.

Let $l\{a, b, c\}$ be an arbitrary permanent axis of rotation ($a \neq 0$, $b \neq 0$, $c \neq 0$).

The components of the angular velocity of the body S_1 along the moving axes are

$$p_0 = \omega a, \quad q_0 = \omega b, \quad r_0 = \omega c \qquad (\omega = \text{const})$$

Let us introduce the notation

$$\left(\frac{\partial U}{\partial \gamma_i}\right)_{\gamma_i = a,b,c} = \beta_i, \quad \left(\frac{\partial^2 U}{\partial \gamma_i^2}\right)_{\gamma_i = a,b,c} = \delta_i, \quad \left(\frac{\partial^2 U}{\partial \gamma_i \partial \gamma_j}\right)_{\gamma_{i,j} = a,b,c} = \varkappa_i \quad \begin{pmatrix} i = 1, 2, 3 \\ j = 1, 2, 3 \end{pmatrix} i \neq j$$

The stability of the permanent rotations will be investigated with respect to the variables p, q, r, γ_1, γ_2, γ_3. Setting in the perturbed motion

$$p = p_0 + \xi_1, \quad q = q_0 + \xi_2, \quad r = r_0 + \xi_3, \quad \gamma_1 = a + \eta_1, \quad \gamma_2 = b + \eta_2, \quad \gamma_3 = c + \eta_3$$

we obtain the equations of the perturbed motion, which permit the following first integrals:

$$\begin{aligned} V_1 = A(\xi_1^2 + 2p_0\xi_1) + B(\xi_2^2 + 2q_0\xi_2) + C(\xi_3^2 + 2r_0\xi_3) - 2\beta_1\eta_1 - 2\beta_2\eta_2 - 2\beta_3\eta_3 - \\ - \delta_1\eta_1^2 - \delta_2\eta_2^2 - \delta_3\eta_3^2 - 2\varkappa_1\eta_1\eta_2 - 2\varkappa_2\eta_2\eta_3 - 2\varkappa_3\eta_3\eta_1 + \ldots = \text{const} \end{aligned} \tag{3.1}$$

$$\begin{aligned} V_2 = A(p_0\eta_1 + \xi_1 a + \xi_1\eta_1) + B(q_0\eta_2 + \xi_2 b + \xi_2\eta_2) + \\ + C(r_0\eta_3 + \xi_3 c + \xi_3\eta_3) + k_1\eta_1 + k_2\eta_2 + k_3\eta_3 = \text{const} \end{aligned}$$

$$V_3 = \eta_1^2 + \eta_2^2 + \eta_3^2 + 2(a\eta_1 + b\eta_2 + c\eta_3) = 0 \tag{3.2}$$

We shall construct the Liapunov function in the form [9]

$$V = V_1 - 2\omega V_2 + \lambda V_3 + {}^1/_4 \nu V_3^2 \tag{3.3}$$

where by (2.2)

$$\lambda = A\omega^2 = \frac{\beta_1 + \omega k_1}{a} = B\omega^2 + \frac{\beta_2 + \omega k_2}{b} = C\omega^2 + \frac{\beta_3 + \omega k_3}{c} \tag{3.4}$$

$$\begin{aligned} V = A\xi_1^2 + B\xi_2^2 + C\xi_3^2 - 2\omega A\xi_1\eta_1 - 2\omega B\xi_2\eta_2 - 2\omega C\xi_3\eta_3 + \mu_1\eta_1^2 + \mu_2\eta_2^2 + \mu_3\eta_3^2 - \\ - 2\varkappa_1'\eta_1\eta_2 - 2\varkappa_2'\eta_2\eta_3 - 2\varkappa_3'\eta_1\eta_3 + \ldots \end{aligned} \tag{3.5}$$

Here

$$\mu_1 = \lambda - \delta_1 - \nu a^2, \quad \varkappa_1' = -2\varkappa_1 + 2\nu ab \qquad (123, abc)$$

The necessary and sufficient conditions for the function V to be positive-definite are, according to the Sylvester criterion, the following inequalities:

$$\mu_1 - \omega^2 A > 0,\ (\mu_1 - \omega^2 A)(\mu_2 - \omega^2 B) - (\varkappa_1')^2 > 0,\ (\mu_1 - \omega^2 A)(\mu_2 - \omega^2 B)(\mu_3 - \omega^2 C) - (\varkappa_1')^2(\mu_3 - \omega^2 C) - (\varkappa_3')^2(\mu_2 - \omega^2 B) - (\varkappa_2')^2(\mu_1 - \omega^2 A) - 2\varkappa_1'\varkappa_2'\varkappa_3' > 0 \quad (3.6)$$

If $U = \alpha_1\gamma_1 + \alpha_2\gamma_2 + \alpha_2\gamma_3$ and $\nu = 0$, then the conditions (3.6) become

$$\lambda - \omega^2 A > 0,\quad \lambda - \omega^2 B > 0,\quad \lambda - \omega^2 C > 0 \quad (3.7)$$

as obtained previously in [7]. If the gyrostatic moment $\mathbf{k}$ is collinear with the vector ω, that is

$$\frac{k_1}{a} = \frac{k_2}{b} = \frac{k_3}{c} = m\omega \quad (3.8)$$

then the conditions of stability can be written in the form

$$m + \frac{\beta_1}{a\omega^2} > 0,\qquad m + \frac{\beta_2}{b\omega^2} > 0,\qquad m + \frac{\beta_3}{c\omega^2} > 0 \quad (3.9)$$

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